

Another Relation Between Approaches to the Schottky Problem

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1. Introduction

The recent extensive work on several approaches to the Schottky problem has produced marked progress on several fronts. At the same time, it has become apparent that there exist very close connections between the various characterizations of Jacobian varieties described in Mumford's classic lectures [M] and the more recent approaches related to the K.P. equation. Some of the most striking results of this kind are to be found in the papers [B-D] and [F]. In the first of these, Beauville and Debarre show that for a principally-polarized abelian variety (p.p.a.v.) (A, Θ) of dimension g , the Andreotti-Mayer condition $\dim \Theta_{Sing} \geq g - 4$ is a consequence of any one of the following hypotheses:

- There are distinct points $x, y, z \in A$ such that $\Theta \cap \Theta_z \subset \Theta_x \cup \Theta_y$,
- The Kummer variety $K(A)$ has a trisecant line, and
- The theta function θ_A satisfies the K.P. equation (in the sense described following the Main Theorem below).

In the second paper cited above, Fay indicates a possible relation between the classical Schottky-Jung relations and the K.P. hierarchy.

However, until now, the approach via double translation manifolds has seemed to be quite different from these other approaches to the Schottky problem. The purpose of this paper is to bring this last approach "into the fold" as it were, and to show precisely how it relates to the approaches via trisecants and flexes of the Kummer variety, and via the K.P. equation. Another relation was pointed out in [L], but we believe that the following theorem gives a much more complete indication of the precise connection.

Main Theorem. *Let (A, Θ) be an indecomposable g -dimensional p.p.a.v. over \mathbf{C} , and let $\psi_A: A \rightarrow \mathbf{P}^{2^g-1}$ be the mapping defined by the linear system of second-order theta functions on A . Assume that the image $K(A) = \psi_A(A)$ has a "curve of flexes," or more precisely that there is an irreducible curve $\Gamma \subset A$ such that for generic $p \in \Gamma$*

$$\Gamma - p \subset V_{Y_p} = \{2t \in A \mid t + Y_p \subset \psi_A^{-1}(\ell) \text{ for some line } \ell \subset \mathbf{P}^{2^g-1}\},$$

where Y_p is the artinian length-3 subscheme $\text{Spec}(\mathcal{O}_{\Gamma,p}/m_{\Gamma,p}^3) - p \subset (A, 0)$. Then Θ is a generalized translation manifold. That is, Θ has a local parametrization of the form:

$$(1.1) \quad z_i = \alpha_i(t_1) + A_i(t_2, \dots, t_{g-1}) \quad i = 1, \dots, g$$

It is known that both the hypothesis and the conclusion of this theorem effectively lead to geometric characterizations of Jacobian varieties among all p.p.a.v. For the hypothesis, this is work of Gunning and Welters. See for example [W]. Thus, in what follows we will refer to the hypothesis that $K(A)$ has a “curve of flexes” as *the Gunning-Welters hypothesis*. The subsequent paper [A-D] shows that this condition is actually much *stronger* than necessary to characterize Jacobians. Indeed, the existence of a jet of V_{Y_p} of sufficiently high order at one point is enough. Naturally our proof will not make use of any of these facts. Instead we will connect the Gunning-Welters hypothesis and the closely related solutions

$$(1.2) \quad u(x, y, t) = 2 \frac{\partial^2}{\partial x^2} \log \theta_A(xU + yV + tW + z_0)$$

$(U, V, W \in \mathbf{C}^g)$ of the K.P. equation

$$(1.3) \quad (3/4)u_{yy} = (u_t - (1/4)(6uu_x + u_{xxx}))_x$$

to a differential-geometric criterion for the existence of parametrizations of the form (1.1) for a hypersurface in \mathbf{C}^g . The proof of this result occupies §§2,3 of the present paper.

In one sense, our result is somewhat disappointing because it indicates that the approach to the Schottky problem via translation manifolds can be subsumed in the other standard approaches. However, even though this is true, this point of view does lead to one further characterization of Jacobian varieties that may be of interest in its own right. Namely, (A, Θ) is a Jacobian if and only if its theta function satisfies a certain system of fifth order PDE (found essentially by eliminating U, V, W from the solution of the K.P. equation given in (1.2)). The equations are unfortunately extremely complicated, but a procedure which describes how they may be constructed may be found in [T]. We hope to return to these equations in a future paper.

2. A Consequence of the Gunning-Welters Hypothesis and the K.P. Equation

We begin by fixing some notation. Let $A \cong \mathbf{C}^g / (\mathbf{Z}^g + \Omega \mathbf{Z}^g)$ be a p.p.a.v. and let

$$\theta_A(z, \Omega) = \sum_{n \in \mathbf{Z}^g} \exp \pi i [n^t \Omega n + 2n^t z]$$

be its Riemann theta function. We will also make use of the related theta functions with characteristics

$$\theta \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} (2z, 2\Omega) = \sum_{n \in \mathbf{Z}^g} \exp 2\pi i [(n + \varepsilon/2)^t \Omega (n + \varepsilon/2) + 2(n + \varepsilon/2)^t z]$$

where ε is any vector of zeroes and ones in \mathbf{Z}^g . It is well-known that these functions form a basis for the linear system of second-order theta functions on A . We denote by $\vec{\theta}_2(z)$ the vector-valued function

$$\vec{\theta}_2(z) = \left(\dots, \theta \begin{bmatrix} \varepsilon \\ 0 \end{bmatrix} (2z, 2\Omega), \dots \right)$$

The mapping ψ_A in the Main Theorem is the projectivization of $\vec{\theta}_2(z)$.

Following the work of Welters [W], we interpret the hypothesis of the Main Theorem as follows. Let $p \in \Gamma$. The mapping $Y_p \rightarrow A$ is induced by a local homomorphism

$$\begin{aligned} \mathcal{O}_{A,0} &\rightarrow \mathbf{C}[t]/(t^3) \\ f &\rightarrow \sum_{i=1}^2 \Delta_i(f)t^i \end{aligned}$$

where the Δ_i are certain differential operators. There are corresponding translation-invariant vector fields D_1, D_2 on A such that formally

$$\exp \left(\sum_j D_j t^j \right) \equiv \sum_k \Delta_k t^k \pmod{t^3}$$

The Gunning-Welters hypothesis is that

$$\text{rank} \left(\vec{\theta}_2 \ \Delta_1 \vec{\theta}_2 \ \Delta_2 \vec{\theta}_2 \right) \leq 2$$

at all points of $\frac{1}{2}(\Gamma - p)$. We begin by noting

Proposition 1. ([W]) *The artinian scheme Y_p (defined by D_1 and D_2) is equal to the second order jet of V_{Y_p} at p .*

This follows from the fact that for indecomposable p.p.a.v. A , the rank of the $2^g \times (g(g+1)/2 + 1)$ matrix

$$\left(\vec{\theta}_2(0) \quad \frac{\partial^2}{\partial z_i \partial z_j} \vec{\theta}_2(0) \right)$$

is equal to $g(g+1)/2 + 1$.

Furthermore, the existence of a third-order jet of Γ at p implies that there exist a vector field D_3 and a scalar $d \in \mathbf{C}$ such that

$$(2.1) \quad 0 = [D_1^4 - D_1 D_3 + \frac{3}{4} D_2^2 + d] \vec{\theta}_2(0)$$

We will write $D_1 = \sum_{i=1}^g U_i \frac{\partial}{\partial z_i}$, $D_2 = \sum_{i=1}^g V_i \frac{\partial}{\partial z_i}$, and $D_3 = \sum_{i=1}^g W_i \frac{\partial}{\partial z_i}$. By the Riemann quadratic theta formula, (2.1) can be rewritten as

$$(2.2) \quad 0 = \theta_{xxxx} \theta - 4\theta_{xxx} \theta_x + 3\theta_{xx}^2 + 4\theta_x \theta_t - 4\theta_{xt} \theta + 3\theta_{yy} \theta - 3\theta_y^2 + 8d\theta^2,$$

where $\theta = \theta(x, y, t) = \theta_A(xU + yV + tW + z_0)$. By a direct computation, this equation is seen to be equivalent to the equation obtained by substituting (1.2) into (1.3).

The key point for us will be that, under the assumption that Γ is one-dimensional, we can actually construct a two-dimensional family of relations (2.2) for each $z_0 \in \mathbf{C}^g$. That is, there is a 2-parameter family of triples U, V, W which yield relations as in (2.2). This dimension count may be explained as follows. We have one parameter for the point $p \in \Gamma$, and a second for the choice of $U \in T_p(\Gamma)$.

The group of transformations

$$\begin{aligned} U &\longrightarrow \lambda U \\ V &\longrightarrow \pm(\lambda^2 V + 2\alpha \lambda U) \\ W &\longrightarrow \lambda^3 W + 3\lambda^2 \alpha + 3\lambda \alpha^2 U \\ d &\longrightarrow \lambda^4 d \end{aligned}$$

leaves (2.1) invariant and acts on our set of triples U, V, W . The quotient projects to a curve Γ' in the projective space \mathbf{P}^{g-1} with homogeneous coordinates (U_1, \dots, U_g) . In other words, Γ' is the image of the projectivized Gauss mapping on Γ . (The relation between Γ' and Γ is the same as the relation between the canonical image of a curve C and the W_1 subvariety of $J(C)$.)

Now, let us specialize to the case $z_0 \in \Theta$ so that $\theta(0, 0, 0) = \theta_A(z_0) = 0$. Setting $x = y = t = 0$ in (2.2), we have

$$(2.3) \quad 0 = -4\theta_{xxx} \theta_x + 3\theta_{xx}^2 + 4\theta_x \theta_t - 3\theta_y^2$$

Furthermore, if we now add the extra condition that $U \neq 0$ is tangent to Θ at z_0 , so that $\theta_x(0, 0, 0) = 0$, then (2.3) reduces to

$$0 = \theta_{xx}^2 - \theta_y^2$$

at $x = y = t = 0$, or

$$(2.4) \quad 0 = \left(\sum_{i,j} \theta_{ij}(z_0) U_i U_j \right)^2 - \left(\sum_i \theta_i(z_0) V_i \right)^2,$$

where we have written $\theta_i = \frac{\partial \theta_A}{\partial z_i}$ and $\theta_{ij} = \frac{\partial^2 \theta_A}{\partial z_i \partial z_j}$.

It is interesting to note that it is precisely this same special case of the K.P. equation that was used by Beauville and Debarre in [B-D] to link several different characterizations of Jacobians. We will use (2.4) in a different way in what follows. The condition that $U \in T_{z_0}(\Theta)$ is that

$$(2.5) \quad \sum_i \theta_i(z_0) U_i = 0$$

As p varies on the curve $\Gamma \subset A$, the point with homogeneous coordinates (U_1, \dots, U_g) varies on the curve $\Gamma' \subset \mathbf{P}^{g-1}$. The equation (2.5) defines the hyperplane $\mathbf{P}T_{z_0}(\Theta) \subset \mathbf{P}^{g-1}$, which will meet the curve Γ' in a finite number of points $q_1(z_0), \dots, q_n(z_0)$. If z_0 varies in a small open set on Θ , we can express the coordinates of any one of these intersection points, say q_1 , as analytic functions of any convenient set of local coordinates on Θ . For simplicity, we will assume that $U_1(z_0) \neq 0$ for all z_0 in our small open set on Θ .

Since $\dim(\Gamma') = 1$, we may also assume (by renumbering if necessary) that after we divide by U_1 to obtain affine coordinates, each of $\tau_2 = U_2/U_1, \dots, \tau_g = U_g/U_1$ can be expressed in terms of τ_2 on Γ' . We have completed the preliminary constructions to prove

Proposition 2. *Let (A, Θ) be an indecomposable p.p.a.v. satisfying the Gunning-Welters hypothesis. Then there exists an analytic function τ_2 on an open subset $X \subset \Theta$ and analytic functions $\tau_3(\tau_2), \dots, \tau_g(\tau_2)$, and $\sigma_1(\tau_2), \dots, \sigma_g(\tau_2)$ such that for all $z_0 \in X$,*

- (i) $\sum_{i=1}^g \theta_i(z_0) \tau_i = 0$, (by convention, here and in the following equation we put $\tau_1 = 1$),
- (ii) $\sum_{i,j} \theta_{ij}(z_0) \tau_i \tau_j + \sum_i \theta_i(z_0) \sigma_i = 0$, and
- (iii) there is an analytic function $\lambda = \lambda(\tau_2)$ such that $\sigma_i = \lambda \cdot \frac{d\tau_i}{d\tau_2}$ for $i = 1, \dots, g$.

Proof. (i) is the condition that $(U_1, \dots, U_g) \in T_{z_0}(\Theta)$, which holds by the argument given before the statement of the Proposition. From (2.4) and (i), we can obtain relations such as (ii) for any $\sigma_i = \pm V_i/U_1^2 + \mu U_i$, where μ is an arbitrary function of τ_2 (independent of i). By Proposition 1, at each point of Γ , U and V span the osculating plane to Γ . Hence, for the local parameter τ_2 on Γ , we have $\frac{dU_i}{d\tau_2} = aV_i + bU_i$ for some functions a, b of τ_2 . We can take

$$\begin{aligned} \sigma_i &= \frac{1}{aU_1} \frac{d\tau_i}{d\tau_2} \\ &= \left(\frac{1}{aU_1^2} \frac{dU_i}{d\tau_2} - \frac{U_i}{aU_1^3} \frac{dU_1}{d\tau_2} \right) \\ &= \frac{1}{U_1^2} V_i + \left(\frac{b}{a} - \frac{1}{aU_1^3} \right) U_i \end{aligned}$$

Then (ii) and (iii) follow from (2.4) and (i). \triangle

We remark that since (2.4) factors as

$$0 = \left(\sum_{i,j} \theta_{ij}(z_0) U_i U_j + \sum_i \theta_i(z_0) V_i \right) \cdot \left(\sum_{i,j} \theta_{ij}(z_0) U_i U_j - \sum_i \theta_i(z_0) V_i \right)$$

there are usually at least *two* systems of functions τ_i, σ_j satisfying the conditions of Proposition 2. For if τ_i, σ_j give one solution of the equations (i) and (ii), then $-\tau_i, -\sigma_j$ give another. This is a consequence of the *symmetry* of the theta divisor.

3. Generalized Translation Manifolds

In this section, we will show that the conclusion of Proposition 2 of §2 implies that the theta-divisor Θ is a generalized translation manifold (see the Main Theorem in §1 and [L]). The following proof is inspired by a similar discussion in [T].

Proposition 3. *Let $H \subset \mathbf{C}^g$ be an analytic hypersurface, defined by the equation $f(z_1, \dots, z_g) = 0$. Assume that there exists an analytic function $\tau_2 = \tau_2(z_1, \dots, z_g)$ on H , and analytic functions $\tau_1 = 1, \tau_i(\tau_2)$ for $i = 3, \dots, g$, and $\sigma_j(\tau_2)$ for $j = 1, \dots, g$ satisfying*

- (i) $\sum_i f_i(z) \tau_i = 0$,
- (ii) $\sum_{i,j} f_{ij}(z) \tau_i \tau_j + \sum_i f_i(z) \sigma_i = 0$ and
- (iii) there exists an analytic function $\lambda = \lambda(\tau_2)$ such that $\sigma_i = \lambda \frac{d\tau_i}{d\tau_2}$ for $i = 1, \dots, g$.

If H is not developable (that is, if the rank of the Gauss map on H is generically $g-1$), then there exists an analytic parametrization

$$(3.1) \quad z_i = \alpha_i(t_1) + A_i(t_2, \dots, t_{g-1}) \quad i = 1, \dots, g$$

for H . Conversely, the existence of a parametrization (3.1) for H implies the existence of τ_i and σ_j satisfying (i), (ii), and (iii).

Proof. The converse is easily seen by substituting (3.1) into the equation of H and differentiating twice with respect to t_1 . From

$$0 = \frac{\partial}{\partial t_1} f(\alpha(t_1) + A(t_2, \dots, t_{g-1}))$$

we obtain

$$0 = \sum_i f_i(z) \alpha'_i(t_1)$$

for all $z \in H$. Differentiating again,

$$0 = \frac{\partial^2}{\partial t_1^2} f(\alpha(t_1) + A(t_2, \dots, t_{g-1}))$$

which yields

$$0 = \sum_{i,j} f_{ij}(z) \alpha'_i(t_1) \alpha'_j(t_1) + \sum_i f_i(z) \alpha''_i(t_1)$$

Hence, by renumbering the coordinates if necessary, we can take $\tau_2 = \alpha'_2/\alpha'_1$ and $\tau_i = \alpha'_i/\alpha'_1$ will all be functions of τ_2 . Then following the same idea as in the proof of Proposition 2, letting

$$\sigma_i = \frac{1}{\alpha'_1} \frac{d\tau_i}{dt_1} = \frac{1}{(\alpha'_1)^3} (\alpha'_1 \alpha''_i - \alpha'_i \alpha''_1)$$

we get a system of functions satisfying (i), (ii), and (iii).

For the direct implication, suppose that a system of functions τ_i, σ_j exists satisfying (i), (ii), and (iii). Consider any one of the submanifolds $K \subset H$ defined by setting $\tau_2 = t_0$ (constant). Let $z_0 = (z_{10}, \dots, z_{g0})$ be an arbitrary point on K , and construct the integral curve $\alpha = \alpha(z_1)$ of the system of ODE

$$\frac{dz_2}{dz_1} = \tau_2, \dots, \frac{dz_g}{dz_1} = \tau_g$$

with initial condition z_0 in \mathbf{C}^g . By condition (i), along α

$$(3.2) \quad \sum_i f_i(\alpha) \frac{d\alpha_i}{dz_1} = \sum_i f_i(\alpha) \tau_i = 0$$

Hence α lies on H (but not on K or any of the other submanifolds $\tau_2 = \text{constant}$).

Differentiating (3.2) with respect to z_1 , along α we have

$$(3.3) \quad 0 = \sum_{i,j} f_{ij}(\alpha) \tau_i \tau_j + \sum_i f_i(\alpha) \frac{d\tau_i}{d\tau_2} \frac{d\tau_2}{dz_1}$$

Subtract (3.3) from (ii) to obtain

$$0 = \sum_{i,j} f_i(\alpha) \left(\sigma_i - \frac{d\tau_i}{d\tau_2} \frac{d\tau_2}{dz_1} \right)$$

By hypothesis (iii), we can write $\sigma_i = \lambda \frac{d\tau_i}{d\tau_2}$ so this last equation becomes

$$(3.4) \quad 0 = \left(\sum_i f_i(\alpha) \frac{d\tau_i}{d\tau_2} \right) \left(\lambda - \frac{d\tau_2}{dz_1} \right)$$

The first factor cannot be zero under our hypotheses since if it were, H would be developable. Indeed from (3.3), we would have $\sum_{i,j} f_{ij}(\alpha) \tau_i \tau_j = 0$ and this implies that the rank of the projectivized Gauss map is $\leq g-2$ everywhere on $X \subset H$, as follows. We express the submanifold K locally as the intersection of $f(z_1, \dots, z_g) = 0$ and $z_g = h(z_1, \dots, z_{g-1})$. Then for constant τ_2 , differentiating (i) with respect to z_1, \dots, z_{g-1} on K we obtain

$$(3.5) \quad \sum_i f_{ij}(z_0) \tau_i = - \sum_i f_{ig}(z_0) \tau_i \frac{\partial h}{\partial z_j} \quad j = 1, \dots, g-1$$

Since $\sum_j \tau_j [\sum_i f_{ij}(z_0) \tau_i] = 0$ by (3.3), substituting from (3.5) we obtain

$$0 = \left[\sum_i f_{ig}(z_0) \tau_i \right] \left[\tau_g - \sum_{j=1}^{g-1} \frac{\partial h}{\partial z_j} \tau_j \right]$$

The second factor here is not zero since α is not tangent to K . Hence the first factor must be zero. This together with (3.5) shows that the projectivized Gauss map on H has rank $\leq g - 2$ at z_0 . Since z_0 was general on K , as z_0 and τ_2 vary, we see that H would be developable if the first factor in (3.4) were zero.

Thus, from (3.4) $\lambda = \frac{d\tau_2}{dz_1}$, and the ODE defining the curve α now take the form

$$\frac{dz_1}{d\tau_2} = \frac{1}{\lambda}, \frac{dz_2}{d\tau_2} = \frac{\tau}{\lambda}, \dots, \frac{dz_g}{d\tau_2} = \frac{\tau_g}{\lambda}$$

Integrating from $\tau_2 = t_0$ to $\tau_2 = t_1$ we obtain a parametrization for α as follows.

$$(3.6) \quad z_i = z_{i0} + \int_{t_0}^{t_1} \frac{\tau_i}{\lambda} d\tau_2$$

($i = 1, \dots, g$). The most important feature of these parametric equations is that the integrand in the second term is a function of τ_2 alone. As a consequence, the integral curves α starting from all points $z_0 \in K$ are *parallel translates* of one another in the ambient \mathbf{C}^g . This also implies that the submanifolds K are parallel translates of one another in \mathbf{C}^g . Thus if we substitute any parametrization $z_{i0} = A_i(t_2, \dots, t_{g-1})$ for K into (3.6), we obtain a parametrization of the form (3.1). The hypersurface generated in this way must coincide with H , since as we noted above each integral curve α is contained in H . \triangle

Combining Propositions 2 and 3 completes the proof of the Main Theorem from §1, since the theta divisor of an indecomposable p.p.a.v. is always non-developable.

In [L], we showed that a p.p.a.v. whose theta divisor has *two* distinct parametrizations of the form (3.1) satisfying certain general position hypotheses is the Jacobian of a non-hyperelliptic curve. It seems highly likely to us that these general position hypotheses are satisfied automatically whenever a p.p.a.v. has two parametrizations of the form (3.1), but we do not have a proof. The idea would be to analyze the “degenerate” generalized double translation manifolds where these general position hypotheses do not hold, and show that they have geometric properties incompatible with those of theta divisors, e.g. rulings, etc.

The remark following Proposition 2 shows in addition that for theta divisors the existence of *one* parametrization (3.1) usually implies the existence of a second. It is reasonable to conjecture that the only time this fails is for hyperelliptic Jacobians, where the extra symmetry of the W_1 subvariety causes the two generically distinct solutions of (2.5) to coincide.

We conclude by mentioning once again that sections 17-19 of Tschebotarow’s paper [T] contain a discussion of a systematic procedure for eliminating the $\tau_i, \sigma_i, \lambda$ from the hypotheses of Proposition 3. The result is a simultaneous system of two fifth order PDE on the defining equation of the hypersurface which are satisfied if and only if the hypersurface is a generalized translation manifold. Applied to theta functions, these PDE should characterize the theta functions of Jacobians.

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